Real and Complex Operator Norms

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Abstract

Real and complex norms of a linear operator acting on a normed complexified space are considered. Bounds on the ratio of these norms are given. The real and complex norms are shown to coincide for four classes of operators:

1. real linear operators from $L_p(\mu_1)$ to $L_q(\mu_2)$, $1 \le p \le q \le \infty$;
2. real linear operators between inner product spaces;
3. nonnegative linear operators acting between complexified function spaces with absolute and monotonic norms;
4. real linear operators from a complexified function space with a norm satisfying $\|\Re x\| \le \|x\|$ to $L_\infty(\mu)$.

The inequality $p \le q$ in Case 1 is shown to be sharp.

A class of norm extensions from a real vector space to its complexification is constructed that preserve operator norms.

1 Introduction

By a normed complexified vector space we mean a triple $(X, X_{\mathbb{R}}, \|\cdot\|_X)$, where

- X and $X_{\mathbb{R}}$ are vector spaces over $\mathbb{C}$ and $\mathbb{R}$, respectively;
- X is the algebraic complexification of $X_{\mathbb{R}}$, i.e. each $x \in X$ can be uniquely written in the form $x = x_1 + ix_2$ with $x_1, x_2 \in X_{\mathbb{R}}$, we set $x_1 := \Re x$, $x_2 := \Im x$;

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- the function $\|\cdot\|_X : X \to [0,\infty)$ is a norm on the complex vector space X, in particular $\|\lambda\, x\|_X = |\lambda|\, \|x\|_X$ for all $\lambda \in \mathbb{C}$, $x \in X$.

Let $(X, X_{\mathbb{R}}, \|\cdot\|_X)$ be a normed complexified space, let $(Y, \|\cdot\|_Y)$ be a normed vector space over $\mathbb{C}$, and $A : X \to Y$ be a $\mathbb{C}$-linear operator. Then two operator norms of A with respect to the norms $\|\cdot\|_X$, $\|\cdot\|_Y$ can be defined depending on whether the supremum of the quotient $\|Ax\|_Y/\|x\|_X$ is taken over all $x \in X \setminus \{0\}$ or over all $x \in X_{\mathbb{R}} \setminus \{0\}$. We distinguish these norms by a superscript $\mathbb{C}$ and $\mathbb{R}$ respectively, i.e. we set

$$\|A\|^{\mathbb{C}}_{X,Y} := \sup_{x\in X\setminus\{0\}} \frac{\|Ax\|_Y}{\|x\|_X}, \qquad \|A\|^{\mathbb{R}}_{X,Y} := \sup_{x\in X_{\mathbb{R}}\setminus\{0\}} \frac{\|Ax\|_Y}{\|x\|_X}.$$

The case that the suprema are infinity is not excluded. Obviously, $\|A\|^{\mathbb{R}}_{X,Y} \le \|A\|^{\mathbb{C}}_{X,Y}$.

In this note we address the question how large the ratio of these operator norms can be and under which conditions they coincide. Our main interest is in the case that Y is the algebraic complexification of a real vector space $Y_{\mathbb{R}}$ and A is a *real operator,* i.e., $A(X_{\mathbb{R}}) \subseteq Y_{\mathbb{R}}$. We discuss the above-mentioned question in detail for inner product spaces and the Banach spaces $L^{\mathbb{F}}_p(\mu) = L^{\mathbb{F}}_p(\Omega, \mathcal{B}, \mu)$, where $p \in [1,\infty]$, μ is a positive measure on a σ-algebra $\mathcal{B}$ of subsets of the set Ω and

$$L^{\mathbb{F}}_p(\mu) = \{\, x : \Omega \to \mathbb{F} \quad | \quad x \text{ is } \mathcal{B}\text{-measurable}, \ \ \|x\|_p < \infty \,\}, \qquad \mathbb{F} = \mathbb{R} \text{ or } \mathbb{C},$$

$$\|x\|_p := \begin{cases} \left(\int_\Omega |x(t)|^p \, d\mu_t\,\right)^{1/p}, & 1 \le p < \infty, \\ \operatorname{esssup}_{t\in\Omega} |x(t)|, & p = \infty. \end{cases}$$

Recall that this definition includes as special cases the finite- and infinite-dimensional l_p spaces. They correspond to $\Omega = \{1, \ldots, n\}$ or $\Omega = \mathbb{N}$ and the counting measure μ satisfying $\mu(\{t\}) = 1$ for all $t \in \Omega$. The associated norms are

$$\|x\|_p = \begin{cases} \left(\sum^n_{t\in\Omega} |x_t|^p\right)^{1/p}, & 1 \le p < \infty, \\ \sup_{t\in\Omega} |x_t|, & p = \infty, \end{cases}$$

where $x = [x_1 \ldots x_n]' \in \mathbb{C}^n$ or $x = (x_j)_{j\in\mathbb{N}} \in l_p$.

Bounds on the ratio $\|A\|^{\mathbb{C}}_{X,Y}/\|A\|^{\mathbb{R}}_{X,Y}$ are discussed in Section 2. We show that it is bounded by 2 whenever the norm on the source space satisfies the condition $\|\Re x\| \le \|x\|$. Tighter bounds are derived for a linear map acting on $L_p(\mu)$. If both the source and the target are inner product spaces, the ratio is shown to be bounded by $\sqrt{2}$.

Section 3 is devoted to classes of operators for which the real and the complex norms coincide. The main result of that section, and of the paper, Theorem 3.1, shows that the norms are equal for all real linear maps from $L_p(\mu_1)$ to $L_q(\mu_2)$, provided that $1 \le p \le q \le \infty$. We discuss three more classes of linear operators whose real and complex norms are equal at the end of Section 3.

The goal of Section 4 is to show that the main result of the paper cannot be improved, that is, to describe counterexamples to Theorem 3.1 in the case $p < q$ $(1 \le p, q \le \infty)$.

Perhaps the simplest example that shows that the real and complex norms of a real linear map may be different can be found already in $\mathbb{C}^2$. Let $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. Then, for every $x = [x_1\ x_2]' \in \mathbb{C}^2 \setminus \{0\}$,

$$\frac{\|Ax\|_1}{\|x\|_\infty} = \frac{|x_1 - x_2| + |x_1 + x_2|}{\max\{|x_1|, |x_2|\}}.$$

It can be checked by straightforward calculation [5] that $\|A\|^{\mathbb{R}}_{\infty,1} = 2$ but $\|A\|^{\mathbb{C}}_{\infty,1} = |1-i|+|1+i| = 2\sqrt{2}$. It turns out that a finite-dimensional variation on this example covers the whole range $p<q$, as discussed in detail in Section 4.

Finally, in Section 5 we construct a class of norm extensions from a real space to its complexification that preserve operator norm.

2 Bounds for $\|A\|^{\mathbb{C}}_{X,Y}/\|A\|^{\mathbb{R}}_{X,Y}$

In this section we give simple bounds for the ratio $\|A\|^{\mathbb{C}}_{X,Y}/\|A\|^{\mathbb{R}}_{X,Y}$.

Proposition 2.1 *Let $(X, X_{\mathbb{R}}, \|\cdot\|_X)$ be a normed complexified space. Then for any normed space $(Y, \|\cdot\|_Y)$ and any linear operator $A : X \to Y$,*

$$\|A\|^{\mathbb{C}}_{X,Y} \;\;\leq\;\; c_X \, \|A\|^{\mathbb{R}}_{X,Y}, \tag{1}$$

where

$$c_X := \sup_{0\neq x\in X} \frac{\|\Re x\|_X + \|\Im x\|_X}{\|x\|_X}.$$

(*a*) *Suppose $\|\cdot\|_X$ satisfies*

$$\|\Re x\|_X \leq \|x\|_X \qquad \text{for all } \; x\in X. \tag{2}$$

Then $c_X \leq 2$.

(*b*) *Suppose the space $X_{\mathbb{R}}$ is endowed with an inner product $\langle\cdot,\cdot\rangle : X_{\mathbb{R}} \times X_{\mathbb{R}} \to \mathbb{R}$. Let $\|\cdot\|_X$ be the norm induced by the complexification of this inner product, i.e. ,*

$$\|x\|^2_X = \langle \Re x, \Re x\rangle + \langle \Im x, \Im x\rangle \qquad \text{for all } \; x\in X. \tag{3}$$

Then $c_X = \sqrt{2}$.

Proof: For all $x\in X$, we have

$$\begin{aligned}
\|Ax\|_Y &= \|A\,\Re x + i\,A\,\Im x\|_Y \\
&\leq \|A\,\Re x\|_Y + \|i\,A\,\Im x\|_Y \\
&= \|A\,\Re x\|_Y + \|A\,\Im x\|_Y \\
&\leq \|A\|^{\mathbb{R}}_{X,Y}\,(\|\Re x\|_X + \|\Im x\|_X).
\end{aligned}$$

Thus

$$\frac{\|Ax\|_Y}{\|x\|_X} \leq \|A\|^{\mathbb{R}}_{X,Y} \frac{\|\Re x\|_X + \|\Im x\|_X}{\|x\|_X} \leq \|A\|^{\mathbb{R}}_{X,Y}\, c_X.$$

This gives inequality (1).
(a). Condition (2) yields

$$\|\Re x\|_X + \|\Im x\|_X = \|\Re x\|_X + \|\Re(ix)\|_X \leq \|x\|_X + \|ix\|_X = 2\|x\|_X.$$

Thus $c_X \leq 2$.
(b). Relation (3) implies that $\|x\|^2_X = \|\Re x\|^2_X + \|\Im x\|^2_X \geq (1/2)(\|\Re x\|_X + \|\Im x\|_X)^2$ for all $x\in X$. Thus $c_X \leq \sqrt{2}$. Let $x = (1+i)x_0$ with $x_0 \in X_{\mathbb{R}}$. Then $\|x\|_X =$

$\sqrt{2}(\|\Re x\|_X + \|\Im x\|_X)$. Thus $c_X \geq \sqrt{2}$. $\square$

A norm $\|\cdot\|_X$ is said to be conjugation-invariant if $\|x\|_X = \|\overline{x}\|_X$ for all $x \in X$, where $\overline{x} := \Re x - i\Im x$. It is easily seen that a conjugation-invariant norm satisfies condition (2). We thus have the following corollary.

Corollary 2.2 *If $\|\cdot\|_X$ is conjugation-invariant then $c_X \leq 2$.*

We give a simple example of a norm that is not conjugation-invariant. It also shows that the constant c_X can be arbitrarily large. Let $r > 0$ and define

$$\|x\|_X = r\,|x_1 + ix_2| + |x_2| \qquad \text{for all } x = [x_1 \;\; x_2]' \in X := \mathbb{C}^2.$$

For $x = [1 \;\; i]'$ we have $\|x\|_X = 1$ and $\|\Re x\|_X = r$. Thus $c_X \geq r$.

In order to determine the constant c_X for the L_p-spaces we need Jensen's inequality in the following form.

Lemma 2.3 **(Jensen's inequality)** *Let $(\Omega, \mathcal{B}, \mu)$ be a measure space. Let $\kappa : [0, \infty) \to \mathbb{R}$ be a convex function. Let $f, w \in L^1_{\mathbb{R}}(\mu)$ be nonnegative functions with $\int_\Omega w(t)\, d\mu_t = 1$. Then*

$$\kappa\left(\int_\Omega f(t)\, w(t)\, d\mu_t\right) \leq \int_\Omega \kappa(f(t))\, w(t)\, d\mu_t.$$

Proof: For the special case $\mu(\Omega) = 1$ and $w(t) \equiv 1$ the proof can be found in [7, Theorem 3.3]. To obtain the full statement define a measure $\tilde{\mu}$ on $\mathcal{B}$ by $\tilde{\mu}(B) := \int_B w(t)\, d\mu_t$. Then $\tilde{\mu}(\Omega) = 1$ and $\int_\Omega f(t)\, d\tilde{\mu}_t = \int_\Omega f(t)\, w(t)\, d\mu_t$ for all nonnegative measurable functions f [7, Theorem 1.29]. Thus the result follows from the special case. $\square$

To each measurable function $x : \Omega \to \mathbb{C}$ we associate a phase function $\phi_x : \Omega \to [-\pi, \pi]$ defined by

$$\phi_x(t) := \begin{cases} \pi, & \Im x(t) = 0, \quad \Re x(t) \leq 0, \\ 2\arctan\left(\frac{\Im x(t)}{\Re x(t) + \sqrt{(\Re x(t))^2 + (\Im x(t))^2}}\right), & \text{otherwise.} \end{cases}$$

Then ϕ_x is a measurable function, and, by elementary trigonometry, $x(t) = e^{i\phi_x(t)}|x(t)|$ for all $t \in \Omega$.

The proposition below gives the constants c_X for the L_p-spaces.

Proposition 2.4 *Let $x \in L^{\mathbb{C}}_p(\Omega, \mathcal{B}, \mu) \setminus \{0\}$ where $(\Omega, \mathcal{B}, \mu)$ is any measure space. Then*

$$\frac{\|\Re x\|_p + \|\Im x\|_p}{\|x\|_p} \leq \begin{cases} \sqrt{2} & \text{if } 1 \leq p \leq 2, \\ 2^{1-1/p} & \text{if } 2 \leq p < \infty, \\ 2 & \text{if } p = \infty. \end{cases} \tag{4}$$

In the first case equality holds if $\Re x = \Im x$. In the second and the third case equality holds if $\|\Re x\|_p = \|\Im x\|_p$ and $(\Re x(t))(\Im x(t)) = 0$ for almost all $t \in \Omega$.

Proof: We only show the estimate (4). The proof of other statements is left to the reader. The case $p = \infty$ is covered by case (a) of Proposition 2.1. Let $1 \le p \le 2$. Then the function $0 \le \xi \mapsto \xi^{2/p}$ is convex. Let $w(t) = |x(t)|^p / \int_\Omega |x(\tau)|^p \, d\mu_\tau$. Then $\int_\Omega w(t)\, d\mu_t = 1$. Applying Jensen's inequality, we obtain

$$\begin{aligned}\frac{\|\Re x\|_p^2 + \|\Im x\|_p^2}{\|x\|_p^2} &= \left(\int_\Omega |\cos(\phi_x(t))|^p\, w(t)\, d\mu_t\right)^{2/p} + \left(\int_\Omega |\sin(\phi_x(t))|^p\, w(t)\, d\mu_t\right)^{2/p} \\ &\le \int_\Omega |\cos(\phi_x(t))|^2\, w(t)\, d\mu_t + \int_\Omega |\sin(\phi_x(t))|^2\, w(t)\, d\mu_t \\ &= 1.\end{aligned}$$

Thus, $\|\Re x\|_p + \|\Im x\|_p \le \sqrt{2}\sqrt{\|\Re x\|_p^2 + \|\Im x\|_p^2} \le \sqrt{2}\,\|x\|_p$.

Let $2 \le p < \infty$. Then $(a^p + b^p)^{1/p} \le (a^2 + b^2)^{1/2}$ for any $a, b \ge 0$. This implies $|\Re x(t)|^p + |\Im x(t)|^p \le |x(t)|^p$ for all $t \in \Omega$. Consequently, $\|\Re x\|_p^p + \|\Im x\|_p^p \le \|x\|_p^p$. By convexity, we also have $\left(\frac{1}{2}\|\Re x\|_p + \frac{1}{2}\|\Im x\|_p\right)^p \le \frac{1}{2}\left(\|\Re x\|_p^p + \|\Im x\|_p^p\right)$. Combining the last two inequalities, we obtain

$$\|\Re x\|_p + \|\Im x\|_p \le 2^{1-1/p}\left(\|\Re x\|_p^p + \|\Im x\|_p^p\right)^{1/p} \le 2^{1-1/p}\|x\|_p.$$

□

3 Cases of equality $\|A\|_{X,Y}^{\mathbb{C}} = \|A\|_{X,Y}^{\mathbb{R}}$

We first state the main result of this paper.

Theorem 3.1 *Let $(\Omega_k, \mathcal{B}_k, \mu_k)$ be measure spaces, $k = 1, 2$. Let $A : L_p^{\mathbb{C}}(\mu_1) \to L_q^{\mathbb{C}}(\mu_2)$ be a linear operator that satisfies $A(L_p^{\mathbb{R}}(\mu_1)) \subseteq L_{\mathbb{R}}^q(\mu_2)$. If $1 \le p \le q \le \infty$, then $\|A\|_{p,q}^{\mathbb{R}} = \|A\|_{p,q}^{\mathbb{C}}$.*

The proof is based on the following lemmas.

Lemma 3.2 *Let $(X, X_{\mathbb{R}}, \|\cdot\|_X)$ and $(Y, Y_{\mathbb{R}}, \|\cdot\|_Y)$ be normed complexified spaces. Let $A : X \to Y$ be a linear operator that satisfies $A(X_{\mathbb{R}}) \subseteq Y_{\mathbb{R}}$. Suppose that the following condition holds: For any $x \in X$, $y \in Y$ with $\|x\|_X = \|y\|_Y = 1$ there exists a $\phi \in [0, 2\pi]$ such that $\|\Re(e^{i\phi}x)\|_X \le \|\Re(e^{i\phi}y)\|_Y$. Then $\|A\|_{X,Y}^{\mathbb{R}} = \|A\|_{X,Y}^{\mathbb{C}}$.*

Proof: The case $A = 0$ is trivial. Let $x \in X$ and suppose $Ax \ne 0$. By assumption, there exists $\phi \in [0, 2\pi]$ such that

$$\left\|\Re\left(e^{i\phi}\frac{x}{\|x\|_X}\right)\right\|_X \le \left\|\Re\left(e^{i\phi}\frac{Ax}{\|Ax\|_Y}\right)\right\|_Y \tag{5}$$

Let $\tilde{x} := \Re(e^{i\phi}x) \in X_{\mathbb{R}}$. The condition $A(X_{\mathbb{R}}) \subseteq Y_{\mathbb{R}}$ yields that $\Re(e^{i\phi}Ax) = A\tilde{x}$. Hence, it follows from (5) that $\|Ax\|_Y / \|x\|_X \le \|A\tilde{x}\|_Y / \|\tilde{x}\|_X$. Thus $\|A\|_{X,Y}^{\mathbb{C}} \le \|A\|_{X,Y}^{\mathbb{R}}$. □

Lemma 3.3 *Let $(X, X_{\mathbb{R}}, \|\cdot\|_X)$ be a normed complexified space. Suppose that $\|\Re x\|_X \le \|x\|_X$ for all $x \in X$. Then for any $x \in X$ the map $\phi \mapsto \|\Re(e^{i\phi}x)\|_X$, $\phi \in \mathbb{R}$, is continuous.*

Proof: This follows from the inequalities

$$|\|\Re(e^{i\phi}x)\|_X - \|\Re(e^{i\phi_0}x)\|_X| \leq \|\Re((e^{i\phi}-e^{i\phi_0})x)\|_X \leq \|(e^{i\phi}-e^{i\phi_0})x\|_X = |e^{i\phi}-e^{i\phi_0}|\,\|x\|_X,$$

which hold for all $\phi, \phi_0 \in \mathbb{R}$ □

Lemma 3.4 *Let $1 \leq p \leq q < \infty$. Then for any $x \in L_p^{\mathbb{C}}(\mu)$ with $\|x\|_p = 1$,*

$$\int_0^{2\pi} \|\Re(e^{i\phi}x)\|_p^q \, d\phi \;\leq\; \int_0^{2\pi} |\cos\phi|^q \, d\phi. \tag{6}$$

Equality holds if $p = q$.

Proof: First note that $\Re(e^{i\phi}x(t)) = \cos(\phi_x(t)+\phi)|x(t)|$, where ϕ_x is the phase function defined in Section 2. Since $q \leq p$, the function $0 \leq \xi \mapsto \xi^{q/p}$ is convex, and due to another assumption of the Lemma, $\int_\Omega |x(t)|^p \, d\mu_t = 1$. Hence, by Jensen's inequality as given in Lemma 2.3, for all $\phi \in \mathbb{R}$,

$$\begin{aligned} \|\Re(e^{i\phi}x)\|_p^q &= \left(\int_\Omega |\cos(\phi_x(t)+\phi)|^p \, |x(t)|^p \, d\mu_t\right)^{q/p} \\ &\leq \int_\Omega |\cos(\phi_x(t)+\phi)|^q \, |x(t)|^p \, d\mu_t. \end{aligned} \tag{7}$$

Combined with Fubini's Theorem, this yields

$$\begin{aligned} \int_0^{2\pi} \|\Re(e^{i\phi}x)\|_p^q \, d\phi &\leq \int_0^{2\pi} \int_\Omega |\cos(\phi_x(t)+\phi)|^q \, |x(t)|^p \, d\mu_t \, d\phi \qquad (8) \\ &= \int_\Omega \int_0^{2\pi} |\cos(\phi_x(t)+\phi)|^q \, |x(t)|^p \, d\phi \, d\mu_t \\ &= \int_\Omega \left(\int_0^{2\pi} |\cos(\phi_x(t)+\phi)|^q \, d\phi\right) |x(t)|^p \, d\mu_t \\ &= \int_\Omega \left(\int_0^{2\pi} |\cos(\phi)|^q \, d\phi\right) |x(t)|^p \, d\mu_t \\ &= \int_0^{2\pi} |\cos(\phi)|^q \, d\phi. \end{aligned}$$

If $p = q$, then equality holds in (7) and in (8). □

For the record, we next give an alternative proof of Lemma 3.4 for the finite-dimensional case. It uses the Hölder norm

$$|\!|\!|f|\!|\!| := \left(\int_0^{2\pi} |f(\phi)|^{q/p} \, d\phi\right)^{p/q}, \qquad f \in \mathcal{C}([0,2\pi],\mathbb{R}).$$

Proof of the finite-dimensional version of Lemma 3.4: Write the components of the vector $x \in \mathbb{C}^n$ in the form $x_t = e^{i\phi_t}|x_t|$, $\phi_t \in [0,2\pi]$, $t = 1,\ldots,n$. Then the

tth component of the vector $\Re(e^{i\phi}x)$ is $|x_t|\cos(\phi_t+\phi)$. Let $f_t(\phi) := |x_t|^p\,|\cos(\phi_t+\phi)|^p$. Then

$$\begin{aligned}
|f_t| &= \left(\int_0^{2\pi} |f_t(\phi)|^{q/p}\,d\phi\right)^{p/q}\\
&= \left(\int_0^{2\pi} |x_t|^q\,|\cos(\phi_t+\phi)|^q\,d\phi\right)^{p/q}\\
&= |x_t|^p\left(\int_0^{2\pi} |\cos(\phi)|^q\,d\phi\right)^{p/q}.
\end{aligned}$$

Thus

$$\begin{aligned}
\left(\int_0^{2\pi} \|\Re(e^{i\phi}x)\|_p^q\,d\phi\right)^{p/q} &= \left(\int_0^{2\pi} (\textstyle\sum_{t=1}^n |x_t|^p\,|\cos(\phi_t+\phi)|^p)^{q/p}\,d\phi\right)^{p/q}\\
&= |\textstyle\sum_{t=1}^n f_t|\\
&\le \textstyle\sum_{k=t}^n |f_t| \qquad (9)\\
&= \|x\|_p^p\left(\int_0^{2\pi} |\cos(\phi)|^q\,d\phi\right)^{p/q}.
\end{aligned}$$

Thus

$$\int_0^{2\pi} \|\Re(e^{i\phi}x)\|_p^q\,d\phi \le \|x\|_p \int_0^{2\pi} |\cos(\phi)|^q\,d\phi. \qquad (10)$$

If $p = q$, then equality holds in (9) and (10). $\square$

We are now in a position to prove Theorem 3.1 for $q < \infty$.

Proof of Theorem 3.1 for $q < \infty$: Let $1 \le p \le q < \infty$. Let $x \in L_{\mathbb{C}}^p(\mu)$, $y \in L_{\mathbb{C}}^q(\mu)$ with $\|x\|_p = \|y\|_q = 1$. Since the inequality $\|\Re(z)\|_p \le \|z\|_p$ holds for all $z \in L_p^{\mathbb{C}}(\mu)$ and all $p \in [1,\infty]$, the function $\phi \mapsto \|\Re(e^{i\phi}y)\|_q^q - \|\Re(e^{i\phi}x)\|_p^q$ is continuous by Lemma 3.3. It follows from Lemma 3.4 that

$$\int_0^{2\pi} (\|\Re(e^{i\phi}y)\|_q^q - \|\Re(e^{i\phi}x)\|_p^q)\,d\phi \;\ge\; 0.$$

Hence, the integrand is nonnegative for at least one $\phi_0 \in [0,2\pi]$. Thus $\|\Re(e^{i\phi_0}y)\|_q \ge \|\Re(e^{i\phi_0}x)\|_p$. Now, Lemma 3.2 yields the result. $\square$

For $q = \infty$, the statement of Theorem 3.1 is covered by the following more general result.

Theorem 3.5 *Let $(\Omega,\mathcal{B},\mu)$ be a measure space. Let $(X, X_{\mathbb{R}}, \|\cdot\|_X)$ be a normed complexified space. Suppose that $\|\Re x\|_X \le \|x\|_X$ for all $x \in X$. Let $A : X \to L_\infty^{\mathbb{C}}(\mu)$ be a linear operator that satisfies $A(X_{\mathbb{R}}) \subseteq L_\infty^{\mathbb{R}}(\mu)$. Then $\|A\|_{X,\infty}^{\mathbb{R}} = \|A\|_{X,\infty}^{\mathbb{C}}$.*

For the proof we need yet another lemma.

Lemma 3.6 *Let $x \in L_{\mathbb{C}}^\infty(\mu)$. Then there is a $\phi_0 \in [0,2\pi]$ with $\|\Re(e^{i\phi_0}x)\|_\infty = \|x\|_\infty$.*

Proof: By Lemma 3.3 the function $\phi \mapsto \|\Re(e^{i\phi}x)\|_\infty$, $\phi \in [0, 2\pi]$, is continuous. It therefore attains its maximum. Hence it is enough to show that to each $0 < c < 1$ there corresponds a $\phi_c \in [0, 2\pi]$ such that $\|\Re(e^{i\phi_c}x)\|_\infty \geq c\,\|x\|_\infty$. Let $M := \{\, t\in\Omega \mid |x(t)| \geq \sqrt{c}\,\|x\|_\infty \,\}$. Note that $\mu(M) > 0$, since $\sqrt{c} < 1$. Choose an integer $n > 1$ such that $\cos(\theta) > \sqrt{c}$ for all $\theta\in\mathbb{R}$ with $|\theta| \leq \frac{\pi}{n}$. Since M is the union of the sets

$$M_\ell = \left\{\; t\in M \;\middle|\; |\phi_x(t) - \frac{2\pi}{n}\ell| \leq \frac{\pi}{n} \;\right\}, \qquad \ell = 1, \ldots, n-1,$$

we have that $\mu(M_\ell) > 0$ for at least one $\ell =: \ell_0$. Let $\phi_c = -\frac{2\pi}{n}\ell_0$. Then for any $t\in M_{\ell_0}$, we have $\cos(\phi_x(t) + \phi_c) > \sqrt{c}$, and hence

$$\Re(e^{i\phi_c}x(t)) \;=\; \cos(\phi_x(t) + \phi_c)\,|x(t)| \geq c\|x\|_\infty.$$

This yields $\|\Re(e^{i\phi_c}x)\|_\infty \geq c\,\|x\|_\infty$. $\square$

Proof of Theorem 3.5: Let $x\in X$, $y\in L_\infty^{\mathbb{C}}(\mu)$ with $\|x\|_X = \|y\|_\infty = 1$. By Lemma 3.6, there is a ϕ_0 with $\|\Re(e^{i\phi_0}y)\|_\infty = 1$. Furthermore, we have $\|\Re(e^{i\phi_0}x)\|_X \leq \|e^{i\phi_0}x\|_X = 1$. Thus $\|\Re(e^{i\phi_0}x)\|_\infty \leq \|\Re(e^{i\phi_0}y)\|_\infty$. Now, the Theorem follows from Lemma 3.2. $\square$

Theorem 3.5 can be proved more directly in a finite-dimensional case. Then the norm on the target space, say $\mathbb{C}^m$, is nothing but a *weighted maximum norm,* i.e., a norm of the form

$$\|y\|_w = \max_{j\in\underline{m}}(w_j\,|y_j|),$$

where $w = [w_1 \ldots w_m]^T > 0$ is a positive vector of weights. The source space $\mathbb{C}^n$, however, does not have to be equipped with a Hölder norm but, more generally, with any *absolute norm,* i.e., a norm that satisfies

$$\|x\| = \|\,|x|\,\| \qquad \text{for all} \qquad x \in \mathbb{C}^n.$$

A n absolute vector norm is necessarily *monotonic* [1]: if $0 \leq x \leq y$ componentwise, then $\|x\| \leq \|y\|$. Hölder norms are obviously absolute.

Proposition 3.7 *Let $\|\cdot\|_\alpha$ be an absolute norm on $\mathbb{C}^n$ and let $\|\cdot\|_w$ be a weighted maximum norm on $\mathbb{C}^m$. Then $\|A\|_{\alpha,w}^{\mathbb{R}} = \|A\|_{\alpha,w}^{\mathbb{C}}$ for every $A = [a_{jk}]\in\mathbb{R}^{m\times n}$.*

Proof: Let $x\in\mathbb{C}^n$ and $j\in\underline{m}$ be such that $\|Ax\|_w = w_j\left|\sum_{k\in\underline{n}} a_{jk}x_k\right|$. Set

$$\widetilde{x}_k := \begin{cases} |x_k|, & a_{jk} \geq 0 \\ -|x_k| & \text{otherwise.} \end{cases}$$

Then we have for the vector $\widetilde{x} = [\widetilde{x}_1 \ldots \widetilde{x}_n]' \in \mathbb{R}^n$ that $\|\widetilde{x}\|_\alpha = \|\,|x|\,\|_\alpha = \|x\|_\alpha$ and $\|A\widetilde{x}\|_w \geq w_j\left|\sum_{j\in\underline{m}} a_{jk}\widetilde{x}_k\right| = w_j\sum_{k\in\underline{n}}|a_{jk}x_k| \geq \|Ax\|_w$. Thus $\|A\widetilde{x}\|_w/\|\widetilde{x}\|_\alpha \geq \|Ax\|_w/\|x\|_\alpha$. Thus $\|A\|_{\alpha,w}^{\mathbb{R}} \geq \|A\|_{\alpha,w}^{\mathbb{C}}$. $\square$

Recall that the *dual norm* $\|\cdot\|_\alpha^d$ associated with a given norm $\|\cdot\|_\alpha$ on $\mathbb{C}^n$ is defined by

$$\|a\|_\alpha^d = \max_{x\in\mathbb{C}^n\setminus\{0\}} \frac{|a'x|}{\|x\|_\alpha} \qquad \text{for } a \in \mathbb{C}^n. \tag{11}$$

By specializing Proposition 3.7 to the case $m = 1$, we obtain the following well-known fact [1].

Corollary 3.8 *Let $\|\cdot\|_\alpha$ be an absolute norm on $\mathbb{C}^n$. If $a\in\mathbb{R}^n$, then the maximum in (11) is attained for a real vector x.*

The next result is a variant of Theorem 3.1 for inner product spaces.

Theorem 3.9 *Let $(X, X_\mathbb{R}, \|\cdot\|_X)$, $(Y, Y_\mathbb{R}, \|\cdot\|_Y)$ be normed complexified space with*

$$\begin{aligned}
\|x\|_X^2 &= \langle \Re x, \Re x\rangle + \langle \Im x, \Im x\rangle && \text{for } x\in X,\\
\|y\|_Y^2 &= (\Re x, \Re x) + (\Im y, \Im y) && \text{for } y\in Y,
\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ and $(\cdot,\cdot)$ are inner products on on $X_\mathbb{R}$ and $Y_\mathbb{R}$ respectively. Then $\|A\|_{X,Y}^\mathbb{R} = \|A\|_{X,Y}^\mathbb{C}$ for any linear operator $A : X \to Y$ satisfying $A(X_\mathbb{R}) \subseteq Y_\mathbb{R}$.

Proof: A straightforward computation yields that for any $x\in X$,

$$\int_0^{2\pi} \|\Re(e^{i\phi}x)\|_X^2 \; d\phi = \left(\int_0^{2\pi} \cos^2(\phi)\, d\phi\right) \|x\|_X^2 = \pi \,\|x\|_X^2.$$

The same relation holds for $y\in Y$ and $\|\cdot\|_Y$. Thus, if $\|x\|_X = \|y\|_Y = 1$,

$$\int_0^{2\pi} (\|\Re(e^{i\phi}y)\|_Y^2 - \|\Re(e^{i\phi}x)\|_X^2)\, d\phi = 0.$$

Hence, by continuity $\|\Re(e^{i\phi}x)\|_X \le \|\Re(e^{i\phi}y)\|_Y$ for some $\phi \in [0, 2\pi]$, and Lemma 3.2 applies. $\square$

We close the section by showing that the real and complex norms of a nonnegative linear operator coincide whenever the source X and the target Y are complexified function spaces where $\|\cdot\|_X$ is absolute and $\|\cdot\|_Y$ is absolute and monotonic. This means that $X_\mathbb{R}$ and $Y_\mathbb{R}$ are spaces of real-valued functions and, for each $x \in X$ ($y \in Y$), the absolute value function $|x| := \sqrt{(\Re x)^2 + (\Im x)^2}$ ($|y| := \sqrt{(\Re y)^2 + (\Im y)^2}$) is an element of the space $X_\mathbb{R}$ ($Y_\mathbb{R}$) and $\|x\|_X = \|\,|x|\,\|_X$ ($\|y\|_Y = \|\,|y|\,\|_Y$). Moreover, if $f, g \in Y_\mathbb{R}$ and $0 \le f \le g$ pointwise, then $\|f\|_Y \le \|g\|_Y$. An operator from X to Y is *nonnegative* if it is real and it maps nonnegative functions from $X_\mathbb{R}$ to nonnegative functions from $Y_\mathbb{R}$.

Theorem 3.10 *Let $(X, X_\mathbb{R}, \|\cdot\|_X)$ and $(Y, Y_\mathbb{R}, \|\cdot\|_Y)$ be complexified function spaces, let $\|\cdot\|_X$ be absolute and $\|\cdot\|_Y$ be absolute and monotonic, and let $A : X \to Y$ be a nonnegative linear operator. Then $\|A\|_{X,Y}^\mathbb{C} = \|A\|_{X,Y}^\mathbb{R}$.*

Proof: First note that the absolute value function is defined by the property

$$|f| = \inf\{g \ge 0 : g \ge \Re(zf) \text{ whenever } z \in \mathbb{C},\ |z| = 1\}.$$

Also observe that A, being a real operator, commutes with taking the real value of a function. Since A is nonnegative nad linear, $Af \geq Ag$ whenever $f \geq g$, hence

$$\Re(zAf) = A\Re(zf) \leq A|f| \quad \text{whenever} \quad z \in \mathbb{C},\ |z| = 1.$$

This implies that $|Af| \leq A|f|$. By the absoluteness and monotonicity of $\|\cdot\|_Y$, we have $\|Af\|_Y = \|\,|Af|\,\|_Y \leq \|\,A|f|\,\|_Y$, whereas $\|f\|_X = \|\,|f|\,\|_X$ since $\|\cdot\|_X$ is absolute. The function $f \in X$ was arbitrary, so the norm $\|A\|^{\mathbb{C}}_{X,Y}$ is in fact equal to

$$\sup_{0 \neq f \in X_R,\ f=|f|} \frac{\|Af\|_Y}{\|f\|_X}$$

and therefore to $\|A\|^{\mathbb{R}}_{X,Y}$. □

Alternatively, in the finite-dimensional case, the equality of real and complex norms can be seen as follows. We denote by $|A|$ the matrix of absolute values of $A \in \mathbb{C}^{m\times n}$.

Proposition 3.11 *Let $\|\cdot\|_\alpha$, $\|\cdot\|_\beta$ be absolute norms on $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively. Then $\|A\|^{\mathbb{C}}_{\alpha,\beta} \leq \|\,|A|\,\|^{\mathbb{R}}_{\alpha,\beta}$ for all $A \in \mathbb{C}^{m\times n}$.*

Proof: The monotonicity property of absolute norms yields that for any $x \in \mathbb{C}^n$,

$$\begin{aligned}
\|Ax\|_\beta &= \|\,|Ax|\,\|_\beta \\
&\leq \|\,|A|\,|x|\,\|_\beta \\
&\leq \|\,|A|\,\|^{\mathbb{R}}_{\alpha,\beta}\,\|\,|x|\,\|_\alpha \\
&= \|\,|A|\,\|^{\mathbb{R}}_{\alpha,\beta}\,\|x\|_\alpha.
\end{aligned}$$

The result follows. □

Corollary 3.12 *If a matrix $A \in \mathbb{R}^{m\times n}$ is elementwise nonnegative and the underlying norms $\|\cdot\|_\alpha$ on $\mathbb{C}^n$ and $\|\cdot\|_\beta$ on $\mathbb{C}^m$ are absolute, then $\|A\|^{\mathbb{C}}_{\alpha,\beta} = \|A\|^{\mathbb{R}}_{\alpha,\beta}$.*

4 Cases of inequality $\|A\|^{\mathbb{C}}_{X,Y} > \|A\|^{\mathbb{R}}_{X,Y}$

We now show that the main result, Theorem 3.1, is sharp already in the finite-dimensional case. In other words, for any $p > q \geq 1$, there exists a real matrix A such that

$$\|A\|^{\mathbb{C}}_{p,q} > \|A\|^{\mathbb{R}}_{p,q}.$$

We begin with the case $p > q$, $q < 2$.

Proposition 4.1 *Let $p > q$, $q < 2$, and let*

$$A := \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}.$$

Then $\|A\|^{\mathbb{C}}_{p,q} > \|A\|^{\mathbb{R}}_{p,q}$.

Proof: First let us show that the value of the real (p,q)-norm of the matrix A is *not* attained at the vector with just one nonzero component. Due to the symmetry of the entries of A, it is enough to argue about the first unit vector $v := [1,0,0]'$. Consider its small real perturbation $v(\varepsilon) := [1,\varepsilon,0]'$ and compare corresponding ratios of norms:

$$
\begin{aligned}
\frac{\|Av\|_q^q}{\|v\|_p^q} &= 4 \\
\frac{\|Av(\varepsilon)\|_q^q}{\|v(\varepsilon)\|_p^q} &= 4 + 2\varepsilon^q - 4\frac{q}{p}\varepsilon^p + q(q-1)\varepsilon^2 + o(\varepsilon^2).
\end{aligned}
$$

The latter expression is strictly greater than 4 for small ε, since $2\varepsilon^q$ is then the smallest order term. Thus, the real (p,q)-norm of A is attained at a vector with at least two nonzero components, say $v_{\min} := [x,y,z]'$. Again, due to the form of the matrix A, the components can be assumed to be all nonnegative and ordered so that $x \geq y \geq z \geq 0$, $y > 0$.

Now, consider the vector $w := [ix,y,z]'$. Since the function $f(x) := x^{q/2}$ is strictly concave on the nonnegative real axis, we have

$$
\begin{aligned}
(x^2+y^2)^{q/2} &> \frac{((x+y)^2)^{q/2} + (x-y)^2)^{q/2}}{2}, \\
(x^2+z^2)^{q/2} &\geq \frac{((x+z)^2)^{q/2} + (x-z)^2)^{q/2}}{2},
\end{aligned}
$$

hence $\|Av_{\min}\|_q^q < \|Aw\|_q^q$, whereas $\|v_{\min}\|_p = \|w\|_p$. Hence, the complex (p,q)-norm of A is strictly bigger than its real norm. □

The case $p > q \geq 2$ reduces to the case we just considered due to duality. We state this formally as a lemma.

Lemma 4.2 *Suppose that a real matrix A satisfies $\|A\|_{p,q}^{\mathbb{C}} > \|A\|_{p,q}^{\mathbb{R}}$. Then its transpose A' satisfies $\|A'\|_{q',p'}^{\mathbb{C}} > \|A'\|_{q',p'}^{\mathbb{R}}$, where $p' := p/(p-1)$, $q' := q/(q-1)$.*

Proof: Follows directly from the fact

$$\|A\|_{p,q}^{\mathbb{F}} = \|A'\|_{q',p'}^{\mathbb{F}},$$

which holds for both $\mathbb{F} = \mathbb{R}$ and $\mathbb{F} = \mathbb{C}$. □

Since $p' < 2$ whenever $p > 2$, this observation enables us to produce counterexamples for the case $p > q \geq 2$ out of couterexamples for the previous case.

Corollary 4.3 *Let $p > q \geq 2$, and let*

$$A := \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{bmatrix}.$$

Then $\|A\|_{p,q}^{\mathbb{C}} > \|A\|_{p,q}^{\mathbb{R}}$.

This finishes our proof that the condition $p \leq q$ in the main theorem of this paper cannot be relaxed.

5 Norm extensions

In this section we provide a class of norm extensions from a real vector space to its complexification which preserve operator norms.

Let ν be a norm on $\mathcal{C}([0,2\pi],\mathbb{R})$, the set of continuous real valued functions on the interval $[0,2\pi]$. Then ν is said to be monotone if $0 \le f(t) \le g(t)$ for all $t \in [0,2\pi]$ implies that $\nu(f) \le \nu(g)$. The norm ν is called shift-invariant if for all $\psi \in \mathbb{R}$, $\nu(f) = \nu(f_\psi)$, where $f_\psi(\phi) := f(\,(\phi+\psi) \bmod 2\pi)$. For instance, the L_p-norms are monotone and shift-invariant.

Let $X_\mathbb{R}$ be a vector space over $\mathbb{R}$ endowed with a norm $\|\cdot\|_{X_\mathbb{R}}$. Let X be the algebraic complexification of $X_\mathbb{R}$. It is then easily seen that for any $x \in X$ the function $\phi \mapsto \|\Re(e^{i\phi}x)\|_{X_\mathbb{R}}$, $\phi \in [0,2\pi]$, is continuous. We denote these functions by $\|\Re(e^{i\cdot}x)\|_{X_\mathbb{R}}$. Let ν be a monotone and shift-invariant norm on $\mathcal{C}([0,2\pi],\mathbb{R})$. For $x \in X$ define

$$\|x\|_X := \nu(\|\Re(e^{i\cdot}x)\|_{X_\mathbb{R}}). \tag{12}$$

Proposition 5.1 *The function $\|\cdot\|_X : X \to \mathbb{R}$ is a norm on X. Suppose ν is normalized so that $\nu(\,|\cos(\cdot)|\,) = 1$. Then $\|x\|_X = \|x\|_{X_\mathbb{R}}$ for all $x \in X_\mathbb{R}$. In other words $\|\cdot\|_X$ is an extension of $\|\cdot\|_{X_\mathbb{R}}$.*

Proof: The triangle inequality for $\|\cdot\|_X$ follows from the triangle inequality for $\|\cdot\|_{X_\mathbb{R}}$ and the monotonicity of ν. The identity $\|\lambda x\|_X = |\lambda|\,\|x\|_X$ for all $\lambda \in \mathbb{C}$ is a consquence of the shift invariance of ν. The rest is obvious. □

In the proposition below $Y_\mathbb{R}$ is a second vector space over $\mathbb{R}$ endowed with a norm $\|\cdot\|_{Y_\mathbb{R}}$ and complexification Y. The norm $\|\cdot\|_Y$ is defined by the same ν, i.e.

$$\|y\|_Y := \nu(\|\Re(e^{i\cdot}y)\|_{Y_\mathbb{R}}) \tag{13}$$

Proposition 5.2 *Let the norms on X, Y be defined as in (12), (13), where ν is monotone and shift-invariant. Then $\|A\|^{\mathbb{C}}_{X,Y} = \|A\|^{\mathbb{R}}_{X,Y}$ for all linear maps $A : X \to Y$ satisfying $A(X_\mathbb{R}) \subseteq Y_\mathbb{R}$.*

Proof: We have,

$$\begin{aligned}
\|Ax\|_Y &= \nu(\|\Re(e^{i\cdot}Ax)\|_{Y_\mathbb{R}}) \\
&= \nu(\|A\Re(e^{i\cdot}x)\|_{Y_\mathbb{R}}) \\
&\le \nu(\|A\|^{\mathbb{R}}_{X,Y}\,\|\Re(e^{i\cdot}x)\|_{X_\mathbb{R}}) \\
&= \|A\|^{\mathbb{R}}_{X,Y}\,\nu(\|\Re(e^{i\cdot}x)\|_{X_\mathbb{R}}) \\
&= \|A\|^{\mathbb{R}}_{X,Y}\,\|x\|_X.
\end{aligned}$$

□

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Postscriptum

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